MATH4240: Stochastic Processes Tutorial 6

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Stationary Distribution

Recall an example in tutorial 4, we use the one-step formula in matrix form to calculate the absorption probability $\rho_{\mathcal{C}_i}(x)$ for irreducible closed set $\mathcal{C}_i$ and $x \in \mathcal{S}_T$. As an example, consider the Markov chain with the following transition matrix

$$P = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 0 & 0 & 2/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/3 & 1/4 & 1/6 & 0 & 1/4 & 0 & 0 \end{pmatrix} \end{array}.$$

It is reducible with $\mathcal{C}_1 = \{1, 3, 6\}$, $\mathcal{C}_2 = \{2, 5\}$, $\mathcal{S}_T = \{4, 7\}$. To simplify the notions, we can regard each $\mathcal{C}_i$ as an absorbing state and define the transition probability $P(x, \mathcal{C}_i) = \sum_{y \in \mathcal{C}_i} P(x, y)$ for $x \in \mathcal{S}_T$.

Then the transition matrix can be written as

$$\widetilde{P} = \begin{array}{c} \begin{array}{cccc} \mathcal{C}_1 & \mathcal{C}_2 & 4 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} \end{array} = \begin{pmatrix} I_2 & 0 \\ S & Q \end{pmatrix}.$$

Let $A = \begin{pmatrix} \rho_{\mathcal{C}_1}(4) & \rho_{\mathcal{C}_2}(4) \\ \rho_{\mathcal{C}_1}(7) & \rho_{\mathcal{C}_2}(7) \end{pmatrix}$. Then one-step formula can be written as $A = QA + S$.
Since $I - Q$ is invertible,

$$A = (I - Q)^{-1} S = \begin{pmatrix} 1/2 & -1/2 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}. \tag{1}$$

Hence $\rho_{\mathcal{C}_1}(4) = \rho_{\mathcal{C}_2}(4) = \rho_{\mathcal{C}_1}(7) = \rho_{\mathcal{C}_2}(7) = 1/2$.

Stationary Distribution

Thus we have

$$\lim_{k\to\infty} \widetilde{P}^k = \begin{array}{c} \begin{matrix} \mathcal{C}_1 & \mathcal{C}_2 & 4 & 7 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix} \end{array}.$$

Now, let's try to compute

$$\lim_{k\to\infty} P^k$$

.

By reordering, we have

$$\bar{P} = \begin{array}{c} \begin{array}{ccccccc} 1 & 3 & 6 & 2 & 5 & 4 & 7 \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 1/3 & 1/6 & 0 & 1/4 & 1/4 & 0 & 0 \end{pmatrix} \end{array}.$$

Write it as

$$\bar{P} = \begin{pmatrix} P_{\mathcal{C}_1} & 0 & 0 \\ 0 & P_{\mathcal{C}_2} & 0 \\ S_1 & S_2 & Q \end{pmatrix}$$

Then,

$$\lim_{k\to\infty} \bar{P}^k = \begin{pmatrix} \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_1 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} \pi_2 \\ \vdots \\ \pi_2 \end{bmatrix} & 0 \\ A_1 & A_2 & 0 \end{pmatrix},$$

where $\begin{bmatrix} \pi_i \\ \vdots \\ \pi_i \end{bmatrix}$ is the stationary distribution for $\mathcal{C}_i$ and $A_i = \begin{bmatrix} \rho_{\mathcal{C}_i}(x_1)\pi_i \\ \vdots \\ \rho_{\mathcal{C}_i}(x_n)\pi_i \end{bmatrix}$

Stationary Distribution

Recall that we have $\pi_i P_{\mathcal{C}_i} = \pi_i$. Thus, to find π_i, we need to solve $\sum_{x \in \mathcal{C}_i} \pi_i(x) = 1$ and

$$(P_{\mathcal{C}_i}^T - I)\pi_i^T = 0$$

with $0 \le \pi_i(x) \le 1$.
Now,

$$P_{\mathcal{C}_1}^T - I = \begin{bmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{bmatrix}.$$

Solve it, we have $\pi_1 = (1/3, 1/3, 1/3)$ and

$$\lim_{k \to \infty} P_{\mathcal{C}_1}^k = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Stationary Distribution

Note that, $\pi_1 = (1/3, 1/3, 1/3)$. Similarly, we have $\pi_2 = (3/7, 4/7)$. Combining the fact that $\rho_{\mathcal{C}_1}(4) = \rho_{\mathcal{C}_1}(7) = \rho_{\mathcal{C}_2}(4) = \rho_{\mathcal{C}_2}(7) = 1/2$, we have

$$\lim_{k\to\infty} \bar{P}^k = \begin{pmatrix} \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_1 \end{bmatrix} & 0 & 0 \\ 0 & \begin{bmatrix} \pi_2 \\ \vdots \\ \pi_2 \end{bmatrix} & 0 \\ A_1 & A_2 & 0 \end{pmatrix} = \begin{array}{c} \begin{array}{ccccccc} 1 & 3 & 6 & 2 & 5 & 4 & 7 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/7 & 4/7 & 0 & 0 \\ 0 & 0 & 0 & 3/7 & 4/7 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 3/14 & 2/7 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 3/14 & 2/7 & 0 & 0 \end{pmatrix} \end{array}.$$

Stationary Distribution

Reordering it, we have

$$\lim_{k\to\infty} P^k = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 3/7 & 0 & 0 & 4/7 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/6 & 3/14 & 1/6 & 0 & 2/7 & 1/6 & 0 \\ 0 & 3/7 & 0 & 0 & 4/7 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/6 & 3/14 & 1/6 & 0 & 2/7 & 1/6 & 0 \end{pmatrix} \end{array}.$$

Stationary Distribution

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$$\lim_{k\to\infty} P^k = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 3/7 & 0 & 0 & 4/7 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/6 & 3/14 & 1/6 & 0 & 2/7 & 1/6 & 0 \\ 0 & 3/7 & 0 & 0 & 4/7 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/6 & 3/14 & 1/6 & 0 & 2/7 & 1/6 & 0 \end{pmatrix} \end{array}.$$

One More Example

Find $\lim_{k\to\infty} P^k$ for

$$P = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{pmatrix} \end{array}.$$

Note that the chain has two irreducible closed set $\mathcal{C}_1 = \{1, 2\}$ and $\mathcal{C}_2 = \{3\}$, and the transient set $\mathcal{S}_T = \{4, 5, 6\}$.

One More Example

Let $P_1 = \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix}$ and let $\pi_1 = (\pi_1^{(1)}, \pi_1^{(2)})$ be the stationary distribution of P_1. Then

$$\begin{cases} \dfrac{1}{3}\pi_1^{(1)} + \dfrac{1}{2}\pi_1^{(2)} = \pi_1^{(1)}, \\ \pi_1^{(1)} + \pi_1^{(2)} = 1. \end{cases}$$

We get $\pi_1 = (\pi_1^{(1)}, \pi_1^{(2)}) = (3/7, 4/7)$. Hence

$$\lim_{k\to\infty} P_1^k = \begin{pmatrix} 3/7 & 4/7 \\ 3/7 & 4/7 \end{pmatrix}.$$

One More Example

Simplify the Markov chain by regarding $\mathcal{C}_1 = \{1, 2\}$ as one absorbing state to get a new transition matrix

$$\widetilde{P} = \begin{array}{c} \begin{array}{ccccc} \mathcal{C}_1 & \mathcal{C}_2 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \end{pmatrix} \end{array} = \begin{pmatrix} I_2 & 0 \\ S & Q \end{pmatrix}$$

If we write $\lim_{k\to\infty} \widetilde{P}^k = \begin{pmatrix} I_2 & 0 \\ A & 0 \end{pmatrix}$,

One More Example

Then

$$A = (I - Q)^{-1} S = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}.$$

Hence

$$\lim_{k \to \infty} \widetilde{P}^k = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 3/4 & 1/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \end{pmatrix}.$$

One More Example

Since $\begin{pmatrix} 3/4 \\ 1/2 \\ 1/4 \end{pmatrix} \pi_1 = \begin{pmatrix} 9/28 & 3/7 \\ 3/14 & 2/7 \\ 3/28 & 1/7 \end{pmatrix}$, finally we have

$$\lim_{k\to\infty} P^k = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 3/7 & 4/7 & 0 & 0 & 0 & 0 \\ 3/7 & 4/7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 9/28 & 3/7 & 1/4 & 0 & 0 & 0 \\ 3/14 & 2/7 & 1/2 & 0 & 0 & 0 \\ 3/28 & 1/7 & 3/4 & 0 & 0 & 0 \end{pmatrix} \end{array}$$